Generating structured networks based on a weight-dependent deactivation mechanism

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(Received 8 January 2005; published 23 June 2005)

Motivated by the degree-dependent deactivation model generating networks with high clustering coefficient [K. Klemm *et al.*, Phys. Rev. E. **65**, 036123 (2002)], a weight-dependent version is studied to model evolving networks. The growth dynamics of the network is based on a naive weight-driven deactivation mechanism which couples the establishment of new active vertices and the weights' dynamical evolution. Both analytical solutions and numerical simulations show that the generated networks possess a high clustering coefficient larger than that for regular lattices of the same average connectivity. Weighted, structured scale-free networks are obtained as the deactivated vertex is target selected at each time step, and weighted, structured exponential networks are realized for the random-selected case.

DOI: 10.1103/PhysRevE.71.066124 PACS number(s): 89.75.Hc, 05.40.−a, 87.23.Kg

I. INTRODUCTION

Complex networks have attracted an increasing interest in the past few years [1]. The main reason is that they play an important role in the understanding of complex behaviors in real world networks, including the structure of language [2], scientific collaboration networks [3], the Internet [4], power grids [5], food webs [6], biological networks [7], etc. The highly heterogeneous topology of these networks is mainly reflected in two characters, the small average path lengths among any two vertices (small-world property) [8] and a power-law distribution (scale-free property), $P(k)\sim k^{-\gamma}$ with $2\leqslant\gamma\leqslant 3$, for the probability that any vertex has k connections to other vertices [9]. Among these flourishing researches, the aging and weight of vertices are of particular interest [5,10–12], which can influence greatly the evolution of the entire network. For instance, in citation networks, papers cease to receive links because their contents are outdated or summarized in review papers, which are then cited instead; and also some famous papers are likely to be cited longer more than those ordinary ones. The developed network models considering the effect of vertices aging and links weight to the growth of the network are the so-called structured scale-free networks [12,13] and weighted evolving networks [5,11,14,15]. The introduction of vertex aging mechanism and link weight evolving mechanism provide us with a profound view on understanding and characterizing realistic complex systems.

Motivated by some previous researches [12,16,17], in the present work, we proposed a simple model to generate structured networks by weight-driven deactivation mechanism which couples the establishment of new active vertices and the weights' dynamical evolution. Both analytical solutions and numerical simulations show that the generated networks possess a high clustering coefficient larger than that for regular lattices of the same average connectivity. Weighted, structured scale-free (WSSF) networks are obtained as the deactivated vertex is target selected at each time step, and weighted, structured exponential (WSE) networks are realized for the random-selected case. For WSSF networks, it was found that many interesting statistical properties (*vertex strengths* and *link weights*) display good right-skewed distribution observed in many realistic systems, while for WSE networks, all the corresponding properties decay with an approximate exponential form.

II. RELATED WEIGHT-DRIVEN EVOLVING MODELS

Weighted networks can be described by a matrix w_{ij} specifying the weight on the edge connecting the vertices i and j, with $i,j=1,\ldots,N$, where N is the size of the network ($w_{ij}=0$ if the vertices i and j are not connected). The strength of the vertex i can be defined as [11,18]

$$s_i=\sum_{j\in\mathcal{V}(i)}w_{ij}, \tag{1}$$

where the sum runs over the set $\mathcal{V}(i)$ of neighbors of i. Recently, Barrat, Barthélemy, and Vespignani (BBV) [14,16] have proposed a model for the evolving of weighted network when new edges and vertices are continuously added into the network while causing dynamic behavior of the weights. Their model starts from an initial number of completely connected vertices, m_0, with a same assigned weight w_0 to each link. At each subsequent time step, the addition of a new vertex n with m_0 edges and corresponding modification in weights are implemented by the following two rules: (i) The new vertex n is attached at random to a previously existing vertex i according to a strength preferential attachment mechanism $s_i/\Sigma_j s_j$, implying that new vertices connect more likely to vertices handling larger weights. (ii) The additional induced increase δ in strength s_i of the ith vertex is distributed among its nearest neighbors $j\in\mathcal{V}(i)$ according to the rule

$$w_{ij}\rightarrow w_{ij}+\delta\frac{w_{ij}}{s_i}. \tag{2}$$

More recently, Pandya [17] argued that this second rule, though it could be just one possibility, does not follow the

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same mechanism as the first rule. For the case of the worldwide airport network, the first rule can be described as "busy airports get busier" according to the dynamics-driven factor s_i. The second rule, however, can be instead described by "busy routes get busier" since the route i to j having more traffic as indicated by w_{ij} would handle larger portion of the induced traffic δ given by $\delta w_{ij}/s_i$. Pandya rewrites Eq. (2) as

$$w_{ij} \rightarrow w_{ij} + \delta \frac{s_j}{\sum k \in \mathcal{V}(i) s_k}, \quad (3)$$

where $\mathcal{V}(i)$ indicates the set of all neighboring airports of i and $k \neq n$. The last term of Eq. (3) indicates that it is more probable that the induced traffic would go towards the airport j handling maximum traffic s_j among the neighboring airports $\mathcal{V}(i)$, which is then consistent with the mechanism of the first rule of BBV [17]. Moreover, one can easily see in the BBV model that $\lim_{t\rightarrow\infty} s_i(t) \rightarrow \infty$, which is not in accordance with most realistic conditions. The deactivation mechanism of vertices of the growing network introduced by Klemm and Eguiluz [12] can well avoid the case of an infinite increasing of the vertex strength.

III. WEIGHT-DEPENDENT DEACTIVATION MODEL

Inspired by the work in Ref. [12] and the statements indicated in the preceding sections, we propose a deactivation model to study the self-organization of weighted evolving networks. The model describes the growth dynamics of a network with directed links. Rather than the degree-dependent deactivation dynamics of the vertices developed in Ref. [12], our model is based on the weight-dependent deactivation dynamics of the vertices, which can be constructed as the following steps.

First, start from an initial seed of m_0 vertices completely connected by undirected links with assigned weight w_0. By k_i' we denote the in-degree of vertex i—i.e., the number of links pointing to vertex i—and by s_i' the total induced strength by in-degree links of vertex i. Each vertex of the network can be in two different states: active or inactive. As the initial condition we let all the m_0 vertices active. At each time step, a new vertex n is added with m_0 links that are attached to the previously existing m_0 active vertices. Each new added link is assigned weight w_0 and induces a total strength increasing $w_0+\delta$ to the linked active vertex. The additional weight δ will be distributed among the out-degree links of the aim vertex according to the rule

$$w_{ij} \rightarrow w_{ij} + \delta \frac{s_j'}{\sum k \in \mathcal{V}(i) s_k'}. \quad (4)$$

For the sake of simplicity we set $w_0=1.0$ and limit ourselves to the case where the introduction of a new incoming link on node i will trigger only local rearrangements of weights of its nearest neighbors. The new added vertex is always in the *active* state first. Remembering that at each time step only m_0 vertices in the network are permitted to be active and all the others are inactive, we will deactivate one of the m_0+1 active vertices after the new active vertex is added to the network. To perform this, we assume that the probability rate P of deactivation decreases with the total induced strength of the vertices. Then the deactivation probability of a vertex i with induced strength s_i' can be written as

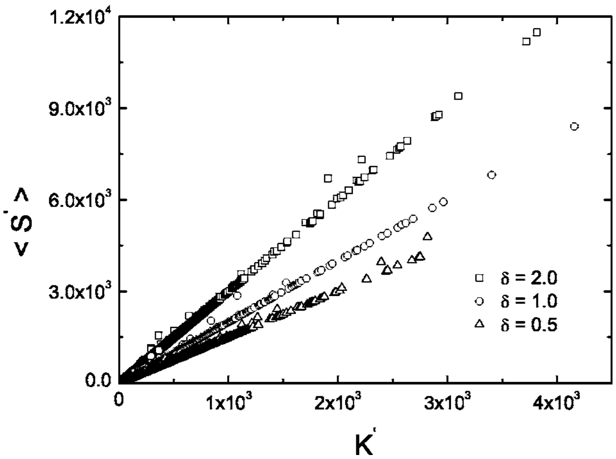

FIG. 1. Illustration of the proportionality between average s' and k' for weighted scale-free networks with parameters $N=10^6$, $w_0=1.0$, $m_0=10$, and $a=m_0(1+\delta)$.

$$P(s_i') \propto \frac{\gamma - 1}{a + s_i'}, \quad (5)$$

where $a > 0$ is a constant bias factor and the normalization factor is defined as $\gamma - 1 = \{\sum_{l \in A}[1/(a+s_l')]\}^{-1}$. The summation runs over the set $\mathcal{A}$ of the currently m_0+1 active vertices. Note that the larger induced strength a vertex possesses, the more difficult for it to be deactivated, or in other words, the easier for it to gain new links.

IV. STRUCTURAL PROPERTIES

A. Weighted scale-free networks

Following Ref. [12], the distribution $N(k')$ of the in-degree k' can be obtained analytically for the model defined above, considering the continuous limit of k'. Let us first derive the distribution $p^{(t)}(k')$ of the in-degree of the active vertices at time t. For $k' > 0$, the time evolution is determined by the master equation

$$p^{(t+1)}(k'+1) = [1 - P(k')]p^{(t)}(k'), \quad (6)$$

where $P(k')$ is the deactivation probability of a vertex with in-degree k'. The boundary value $p(0)$ is a constant reflecting the constant rate of new vertices with initial $k'=0$. To do further investigation, we first get the relation between s' and k' of the vertices from numerical simulations. In Fig. 1, we plot s' as a function of k'. In order to reduce statistical error, the induced strengths of the vertices are calculated as an average and the data are averaged over ten network realizations. The best linear fit gives $s' \simeq (1+\delta)k'$; then, we obtain $P(k') \simeq (1+\delta)P(s')$, where $P(s')$ is the deactivation probability of a vertex with strength s'. Substituting them into Eq. (6) yields

$$p^{(t+1)}(k'+1)=\left(1-\frac{\gamma-1}{\frac{a}{1+\delta}+k'}\right)p^{(t)}(k'). \tag{7}$$

The subsequent thing is just to follow the analytical method in Ref. [12]. Assuming that the fluctuations of the normalization $\gamma-1$ are small enough, such that γ may be treated as a constant, the stationary case $p^{(t+1)}(k')=p^{(t)}(k')$ of Eq. (7) yields

$$p(k'+1)-p(k')=-\frac{\gamma-1}{\frac{a}{1+\delta}+k'}p(k'). \tag{8}$$

Treating k as continuous we write

$$\frac{\mathrm{d}p}{\mathrm{d}k'}=-\frac{\gamma-1}{\frac{a}{1+\delta}+k'}p(k') \tag{9}$$

and obtain the solution

$$p(k')=b\left(\frac{a}{1+\delta}+k'\right)^{-\gamma+1}, \tag{10}$$

with appropriate normalization constant b. In case the total number n of vertices in the network is large compared with the number m_0 of active vertices, the overall in-degree distribution $N(k')$ can be approximated by considering the inactive vertices only. Thus $N(k')$ can be calculated as the rate of change of the degree distribution $p(k')$ of the active vertices. We find

$$N(k')=-\frac{\mathrm{d}p}{\mathrm{d}k'}=c\left(\frac{a}{1+\delta}+k'\right)^{-\gamma}, \tag{11}$$

with $c=(\gamma-1)[a/(1+\delta)]^{\gamma-1}$. The exponent γ is obtained from a self-consistency condition obtained from the average connectivity

$$m_0=c\int_0^\infty \frac{k'}{\left(\frac{a}{1+\delta}+k'\right)^{\gamma}}dk', \tag{12}$$

which gives

$$\gamma=2+\frac{a}{m_0(1+\delta)}. \tag{13}$$

Thus the exponent γ depends only on the ratio $a/m_0(1+\delta)$. If we choose the value of the constant bias $a=m_0(1+\delta)$, Eq. (11) is none other than the probability distribution of vertices total degree $k=(m_0+k')$ of the network. Substituting $a=m_0(1+\delta)$ and $\gamma=3$ into Eq. (11), we get

$$N(k)=\frac{2m_0^2}{k^3}. \tag{14}$$

In Fig. 2, we plot the total degree distribution of the networks with different values of $\delta=0.0,0.5,1.0,2.0$, $m_0=10,7,5$, and $a=m_0(1+\delta)$. A power-law distribution $P(k)\sim(k)^{-\gamma}$ with best-fitted exponent $\gamma=2.96\pm0.05$ is obtained,

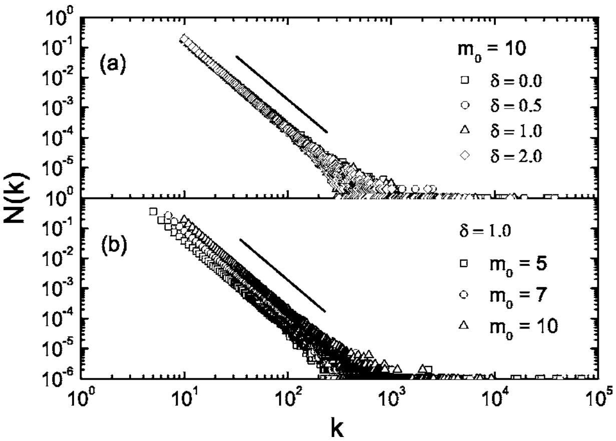

FIG. 2. Illustration of the probability distribution of vertex degrees for weighted scale-free networks with parameters $N=10^6$, $w_0=1.0$, and $a=m_0(1+\delta)$. The straight lines correspond to a power law with exponent $\gamma=3.0$.

which is well in agreement with our analytic result, Eq. (13). In fact the distribution follows a power-law decay but with an exponent γ that depends on m_0, which has also been found in the degree-dependent deactivation model [13]. In order to show the asymptotic power-law behavior of the degree distribution, in Fig. 3, we report the behavior of the exponent γ as a function of m_0. Even for values of $m_0\leqslant 10$, the degree exponents fast approach the limit of large values of m_0.

Notice that each new link added to the network will induce a $w_0+\delta$ strength increase in the aim vertex, which indicates that the vertices with larger in-degree would likewise have larger induced strength. According to the evolving rules, the vertices with larger induced strength have more probability to gain new links; then, the usual degree preferential attachment is reasonably recovered. This means that the right-skewed character of the probability distribution of the interesting statistical properties of the network, such as the vertices total strength, also of the link weights, will re-

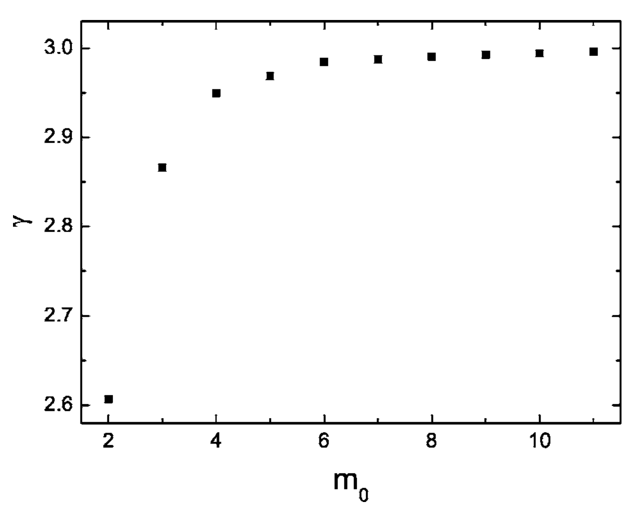

FIG. 3. The value of the exponent γ as a function of m_0 obtained from numerical simulations for weighted scale-free networks with parameters $N=10^6$, $w_0=\delta=1.0$, and $a=m_0(1+\delta)$.

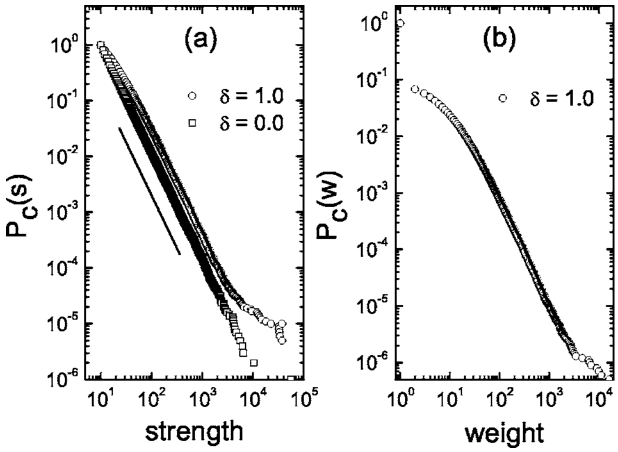

FIG. 4. Cumulative probability distribution of vertex strengths (a) and cumulative link weights (b) for a weighted scale-free network with parameters $N=10^6$, $m_0=10$, $w_0=\delta=1.0$, and $a=m_0(1+\delta)$. By comparison, the case of $\delta=0.0$ is also reported in (a), which recovers the usual degree-dependent deactivation model and the strength decays with a power-law form.

tain. In order to decrease the statistical fluctuation, we report the cumulative probability distribution of the these two properties in Fig. 4. Although distinctly deviating from a simple power-law behavior, the results are well expected, showing good right-skewed character, which is reasonably in agreement with the condition of many realistic systems [3,5,6,9].

B. Weighted exponential networks

We have investigated the case that the constant bias factor a in Eq. (5) is selected as $m_0(1+\delta)$, which gives rise to a power-law decay of the degree distribution. Now we consider the limit case that $a\to\infty$. For the sake of simplicity, we choose $a=Nm_0(1+\delta)$. In the $N\to\infty$ limit the deactivation probability $P(k')$ is independent of k' and δ; i.e., each of the m_0+1 active vertices will be random deactivated with the same probability $1/(m_0+1)$. Then Eq. (8) can be written as

$$p(k'+1)-p(k')=-\frac{1}{m_0+1}p(k'). \tag{15}$$

Again treating k' as continuous we write

$$\frac{dp}{dk'}=-\frac{1}{m_0+1}p(k') \tag{16}$$

and obtain the solution

$$p(k')=b\exp\left(\frac{-k'}{m_0+1}\right), \tag{17}$$

with appropriate normalization constant b. Again, the overall in-degree distribution $N(k')$ can be obtained by

$$N(k')=-\frac{dp}{dk'}=c\exp\left(\frac{-k'}{m_0+1}\right), \tag{18}$$

with normalized fator $c=1/(m_0+1)$. To obtain the total degree distribution, we only need to rewrite Eq. (18) as

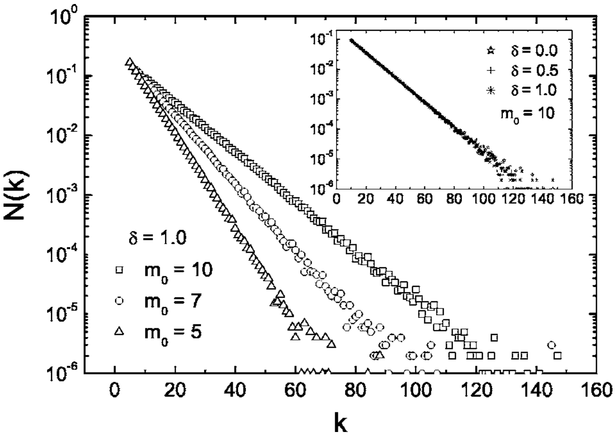

FIG. 5. Illustration of the probability distribution of vertex degrees for weighted exponential networks with parameters $N=10^6$, $w_0=1.0$, and $a=Nm_0(1+\delta)$.

$$N(k)=c\exp(-\gamma k)=\frac{1}{m_0+1}\exp\left(\frac{m_0}{m_0+1}\right)\exp\left(-\frac{k}{m_0+1}\right), \tag{19}$$

where $k=k'+m_0$. Thus, in the $a\to\infty$ limit, weighted networks with exponential decay of the degree distribution are obtained and the exponent γ depends strongly on m_0—i.e., $\gamma\sim 1/(m_0+1)$. In Fig. 5, we plot the total degree distribution of the networks with different values of $\delta=0.0,0.5,1.0$, $m_0=10,7,5$, and $a=Nm_0(1+\delta)$. As expected, $N(k)$ decays exponentially. The behavior of the exponent γ as a function of m_0+1 is given in Fig. 6, and in the inset the data are reported as a log-log representation showing that $\gamma\sim(m_0+1)^{-\beta}$. The best linear fit gives $\beta=1.0\pm0.05$, which is well in agreement with the analytical result. The numerical results of the cumulative probability distribution of the vertex total strength and the link weights for different values of δ are summarized in

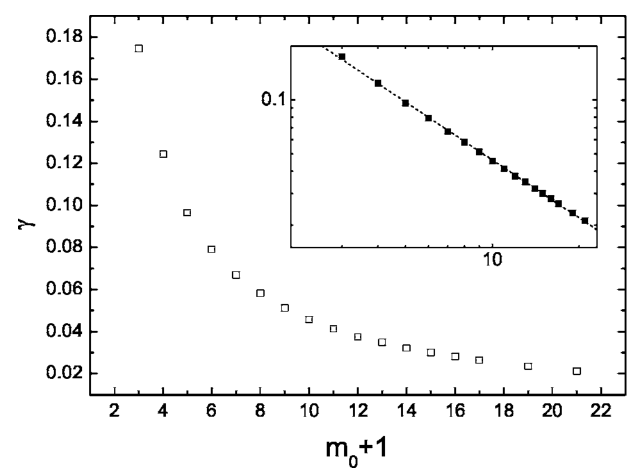

FIG. 6. The value of the exponent γ as a function of m_0+1 obtained from numerical simulations for weighted exponential networks with parameters $N=10^6$, $w_0=\delta=1.0$, and $a=Nm_0(1+\delta)$. In the inset, the date reported on the log-log representation shows that $\gamma\sim(m_0+1)^{-\beta}$, and the best linear fit gives $\beta=1.0\pm0.05$.

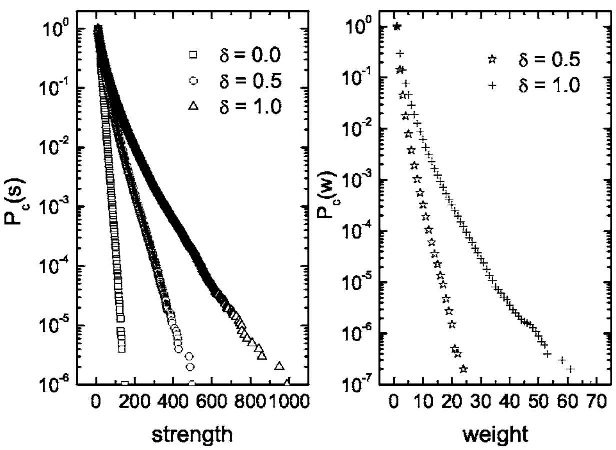

FIG. 7. Cumulative probability distribution of vertex strengths (a) and cumulative link weights (b) for weighted exponential networks with parameters $N=10^6$, $m_0=10$, $w_0=1.0$, and $a=Nm_0(1+\delta)$.

Fig. 7. Similar to the case of weighted scale-free networks, both $P_c(s)$ and $P_c(w)$ deviate from a simple exponential decay behavior for these weighted exponential networks.

C. Average clustering coefficient

To complete our study of the model, we finally study another fundamental topological feature, the average clustering coefficient measuring the average probability with which two neighbors of a vertex are also neighbors to each other. It has been found that many real-world networks display a high clustering coefficient [1]. For the degree-dependent deactivation model, Klemm and Eguiluz have studied the case that the average clustering coefficient depends strongly only on the overall degree distribution $N(k)$ (see details in Ref. [19]), which can be expressed as

$$C=\int_{m_0}^{\infty}\left(1-\frac{(k-m_0+1)(k-m_0)}{k(k-1)}\right)N(k)dk. \qquad (20)$$

The expression is also suitable for the weight-dependent deactivation model we studied here. Thus, in the case of weighted scale-free networks where $N(k)=2m_0^2k^{-3}$ when the constant bias factor $a=m_0(1+\delta)$, we get

$$C=\frac{5}{6}-\frac{7}{30m_0}+\mathcal{O}(m_0^{-2}), \qquad (21)$$

identical to the result in Ref. [19]. In the limit of large m_0 the average clustering coefficient is 5/6. In the case of weighted exponential networks when the constant bias factor $a\to\infty$, inserting Eq. (19) into Eq. (20) and performing the integral over k, we obtain

$$C=\frac{m_0-m_0^2}{m_0+1}\exp\left(\frac{m_0}{m_0+1}\right)f\left(\frac{-m_0}{m_0+1}\right)+\frac{m_0^2-3m_0+2}{m_0+1}\exp\left(\frac{m_0-1}{m_0+1}\right)f\left(\frac{-m_0+1}{m_0+1}\right),$$

where $f(x)$ is a special exponential integral function and has the form $f(z)=-\int_{-z}^{\infty}[\exp(-x)/x]dx$. In the limit of large m_0 the average clustering coefficient gets close to an asymptotic value of 0.789. These results have also been confirmed by extensive numerical simulations (see Fig. 8). The dependence of the average clustering coefficient C on the size N of the weighted networks is reported in Fig. 9, which also shows a similar asymptotic behavior.

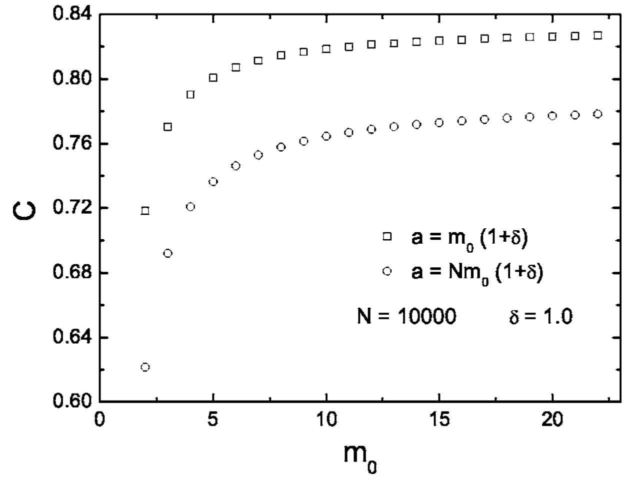

FIG. 8. Illustration of the average clustering coefficient C as a function of m_0. The squares and circles correspond to weighted, structured scale-free networks and weighted, structured exponential networks, respectively.

It is worth noting that the average clustering coefficient of both weighted scale-free networks and weighted exponential networks is higher than that for the corresponding one-dimensional regular lattices whose value is 3/4 in the limit case. Thus the weight-dependent deactivation model generates networks with high clustering—i.e., from weighted, structured scale-free networks to weighted, structured exponential networks depending on the value of the constant bias factor a from $m_0(1+\delta)$ to ∞.

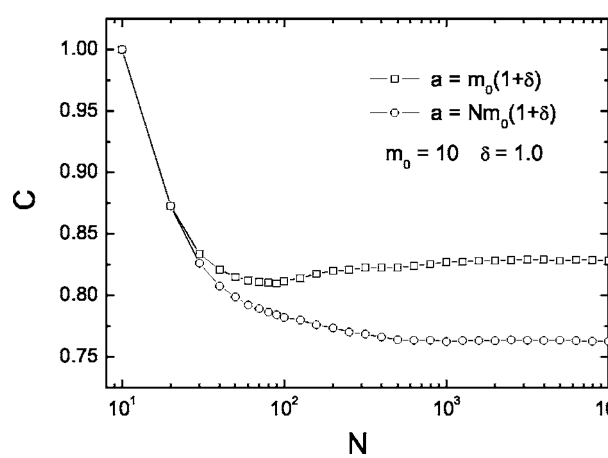

FIG. 9. Illustration of the average clustering coefficient C as a function of N. The squares and circles correspond to weighted, structured scale-free networks and weighted, structured exponential networks, respectively.

V. CONCLUSIONS

In summary, we have studied a simple evolving model for weighted, structured networks. The growth dynamics of the network is governed by a naive weight-driven deactivation mechanism. The deactivation probability is proportional to the inverse of the vertex strength induced by its in-degree links, which characterize the vertices' capability of obtaining further links. By tuning the value of the constant bias factor in the deactivation probability, we have found many interesting evolutionary results of the model. If the value of a is selected appropriately as $m_0(1+\delta)$—i.e., the active vertices are target selected to deactivate with Eq. (5)—the model leads to a power-law probability distribution for the total degree characterized by an exponent $\gamma=3.0$. Besides the vertex degree, some statistical properties of the generated network, such as vertex strength and link weight, display a good right-skewed distribution character, which has been found to be very common in most realistic systems. However, as the value of a tends to ∞—namely, the active vertices are random selected to deactivate with the same probability $1/(m_0+1)$—the model gives rise to an exponential probability distribution for the total degree characterized by an exponent γ which depends strongly on m_0. Particularly, in the limit of large m_0, the weight-dependent deactivation model generates structured networks with high clustering-coefficient values larger than those for the corresponding one-dimensional regular lattices in the limit case.

ACKNOWLEDGMENTS

This work was partly supported by the Natural Science Foundation of Gansu Province (Grant No. ZS011-A25-004-2) and by Lanzhou University.

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